

# Classical limits of quantum mechanics on a non-commutative configuration space

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We consider a model of non-commutative Quantum Mechanics given by two harmonic oscillators over a non-commutative two dimensional configuration space. We study possible ways of removing the non-commutativity based on the classical limit context known as anti-Wick quantization. We show that removal of non-commutativity from the configuration space and from the canonical operators are not commuting operations.

## I. INTRODUCTION

Since the beginning, the classical limit of quantum mechanics has been of primary interest. The most suitable context where to study it is provided by the notion of coherent states as in [1].

In this work, we study the classical limit not of a standard quantum system, but of two quantum harmonic oscillators whose spatial coordinates are themselves non-commuting operators with non-commutative parameter  $\theta$  [2]. One immediately has various possibilities to go to the limit of classical harmonic oscillators on the commutative configuration space  $\mathbb{R}^2$ . One can first go from the non-commutative configuration space to  $\mathbb{R}^2$  by letting  $\theta \rightarrow 0$  and then remove the quantumness by letting  $\hbar \rightarrow 0$ ; one can remove quantumness first and get to a non-quantum system over a non-commutative configuration space and then remove the residual non-commutativity; finally, one can remove both non-commutativities together.

In order to study these possibilities, we use the quantization/de-quantization schemes known as anti-Wick quantization [3]. In such a scheme we first quantize a  $C^*$  algebra of continuous functions with identity by means of suitably constructed Weyl operators and corresponding Gaussian states that allow to set up a positive unital map from functions to bounded operators and then de-quantize it by getting back to functions via another positive unital map. Combining them together one has a means to first let  $\theta \rightarrow 0$  and then  $\hbar \rightarrow 0$  and vice versa: the main result is that the two procedures do not commute.

Further, we study a harmonic like dynamics of the two non-commutative quantum oscillators and show that the asymmetry in the two limits is even stronger; letting  $\theta \rightarrow 0$  first regains the standard quantum mechanics of two independent harmonic oscillators. However, letting  $\hbar \rightarrow 0$  first does not leave any dynamics on the non-quantum system over the non-commutative configuration space.

We start by giving a brief review of the anti-Wick quantization in Section II. Then, in Section III, we briefly recall the model of non-commutative quantum harmonic oscillators in two dimensions; in III A we construct the Weyl operators and Gaussian states on which the anti-Wick quantization is based and in Section IV we study the various classical limits. The time evolution and its classical limits will be discussed in Section V.

## II. ANTI-WICK QUANTIZATION

In this section we shall shortly review the classical limit of quantum mechanics in the algebraic setting known as *anti-Wick quantization*; this technique is based on the quasi-classical properties of *coherent states* whose definition and properties we shall also summarize. For later extension to the non-commutative quan-

tum mechanical context, we shall consider the standard setting of a classical system with  $s$  degrees of freedom described by a phase-space  $\mathbb{R}^{2s}$  with canonical coordinates and momenta  $r = (q, p) \in M$ ;  $q$  and  $p$  denote vectors in  $\mathbb{R}^s$  whose components satisfy the canonical Poisson-bracket relations  $\{q_i, p_j\} = \delta_{ij}$ . From now, scalar products will be denoted by  $(q, p) = \sum_{j=1}^s q_j p_j$  and by  $\|r\|^2$  norm of vectors.

Let  $\hat{r} = (\hat{q}, \hat{p})$  be the  $2s$ -dimensional vector of quantized coordinates and momenta operators acting on the Hilbert space  $\mathcal{H}$  of square-summable functions over  $\mathbb{R}^s$ . They satisfy the Heisenberg commutation relations  $[\hat{r}_i, \hat{r}_j] = i\hbar\Omega_{ij}$ , where  $\Omega$  is the  $2s \times 2s$  symplectic matrix

$$\Omega = \begin{pmatrix} 0 & 1_{s \times s} \\ -1_{s \times s} & 0 \end{pmatrix}, \quad 1_{s \times s} \text{ the } s \times s \text{ identity matrix.} \quad (1)$$

A useful  $C^*$  algebraic description of the quantized system in terms of bounded operators on  $\mathcal{H}$  makes use is of the unitary Weyl operators

$$\hat{W}_\hbar(r) = \exp\left(\frac{i}{\hbar}(r, \Omega \hat{r})\right) = \exp\left(\frac{i}{\hbar}((q, \hat{p}) - (p, \hat{q}))\right), \quad (2)$$

where  $(\cdot, \cdot)$  denotes the scalar product over generic  $2s$ -dimensional vectors. They satisfy the Weyl algebraic relations

$$\hat{W}_\hbar(r_1)\hat{W}_\hbar(r_2) = \exp\left(\frac{i}{2\hbar}(r_2, \Omega r_1)\right) \hat{W}_\hbar(r_1 + r_2), \quad (3)$$

whence they linearly span an algebra whose norm closure known as *Weyl algebra*. One can pass from a real formulation whereby the Weyl operators are labelled by  $r \in \mathbb{R}^{2s}$  to a complex formulation where they are labelled by a complex vector  $z \in \mathbb{C}^s$ : this is done by introducing creation and annihilation operators

$$\hat{a} = \sqrt{\frac{1}{2\alpha\hbar}} \hat{q} + i\sqrt{\frac{\alpha}{2\hbar}} \hat{p}, \quad \hat{a}^\dagger = \sqrt{\frac{1}{2\alpha\hbar}} \hat{q} - i\sqrt{\frac{\alpha}{2\hbar}} \hat{p}, \quad (4)$$

where  $\alpha$  is a suitable parameter such that  $\hat{a}^\#$  is  $a$ -dimensional and  $[\hat{a}_i, \hat{a}_j^\dagger] = \delta_{ij}$ . Then, one rewrites

$$\hat{W}_\hbar(r) = \exp\left(z_r \hat{a}^\dagger - z_r^* \hat{a}\right) =: \hat{W}_\hbar(z_r), \quad z_r = -\frac{q}{\sqrt{2\alpha\hbar}} - ip\sqrt{\frac{\alpha}{2\hbar}}. \quad (5)$$

Let  $|0\rangle_\hbar$  denote the state annihilated by all operators  $\hat{a}_j$ :  $\hat{a}_j|0\rangle_\hbar = 0$ . We shall refer to it as to the ground state, which in position representation amounts to the Gaussian state

$$\langle q|0\rangle_\hbar = \psi_0(q) = \frac{1}{(2\pi\alpha\hbar)^{s/4}} \exp\left(-\frac{\|q\|^2}{2\alpha\hbar}\right). \quad (6)$$

The coherent states

$$|z_r\rangle_\hbar = \hat{W}_\hbar(z_r)|0\rangle_\hbar = e^{-\|z_r\|^2/2} e^{z_r \hat{a}^\dagger} |0\rangle_\hbar \quad (7)$$

are eigenstates of the vector operator  $\hat{a}$  with eigenvalue  $z_r \in \mathbb{C}^s$ ,

$$\hat{a}_j |z_r\rangle_\hbar = z_r^j |z_r\rangle_\hbar = \left(\frac{q_j}{\sqrt{2\alpha\hbar}} + ip_j \sqrt{\frac{\alpha}{2\hbar}}\right) |z_r\rangle_\hbar, \quad (8)$$

whence

$${}_h\langle 0|\hat{W}_\hbar(z_r)|0\rangle_\hbar = e^{-\|r\|_{\alpha, \hbar}^2}, \quad \text{where} \quad \|r\|_{\alpha, \hbar}^2 = \frac{1}{4\alpha\hbar} \|q\|^2 + \frac{\alpha}{4\hbar} \|p\|^2. \quad (9)$$

In order to set a useful algebraic setting for the classical limit, we will consider the  $C^*$  algebra of continuous functions over  $\mathbb{R}^{2s}$  which vanish at infinity to which we add the identity function: we shall denote by  $C_\infty$  this commutative  $C^*$  algebra. In this context, a particularly suitable algebraic setting for the classical limit  $\hbar \rightarrow 0$  is the so-called *anti-Wick* quantization that is based on the over-completeness of coherent states:

$$\hat{1} = \frac{1}{(2\pi\hbar)^s} \int_{\mathbb{R}^{2s}} dr |z_r\rangle_\hbar {}_h\langle z_r|, \quad (10)$$

where  $\hat{1}$  denotes the identity operator on  $\mathcal{H}$ .

Then, one may define two positive maps: a quantization map  $\gamma_{\hbar,0} : C_\infty(\mathbb{R}^s) \mapsto \mathcal{W}_\hbar$ , given by

$$C_\infty(\mathbb{R}^{2s}) \ni F \mapsto \gamma_{\hbar,0}[F] =: \hat{F}_\hbar \in \mathcal{W}_\hbar, \quad \hat{F}_\hbar = \frac{1}{(2\pi\hbar)^s} \int_{\mathbb{R}^{2s}} dr F(r) |z_r\rangle_{\hbar\hbar} \langle z_r|, \quad (11)$$

which represents the quantization of the classical function  $F(r) \in C_\infty(\mathbb{R}^{2s})$ , and a de-quantization map  $\gamma_{0,\hbar} : \mathcal{W}_\hbar \mapsto C_\infty(\mathbb{R}^{2s})$  given by

$$\mathcal{W}_\hbar \ni \hat{X} \mapsto \gamma_{0,\hbar}[\hat{X}] \in C_\infty(\mathbb{R}^{2s}), \quad X(r) = \hbar \langle z_r | \hat{X} | z_r \rangle_\hbar, \quad (12)$$

which *de-quantizes* the operator  $\hat{X}$  mapping it back to a function in  $C_\infty(\mathbb{R}^{2s})$ .

**Remark 1** *The quantization and de-quantization maps are positive as they send positive functions into positive operators and vice versa; they are unital as they map the identity function in  $C_\infty(\mathbb{R}^{2s})$  into the identity operator  $\hat{1} \in \mathcal{W}_\hbar$ .*

Now, one computes

$$\begin{aligned} \gamma_{0,\hbar} \circ \gamma_{\hbar,0}[F](r) &= \frac{1}{(2\pi\hbar)^s} \int_{\mathbb{R}^{2s}} dr' F(r') |\hbar \langle z_r | z_{r'} \rangle_\hbar|^2 = \frac{1}{(2\pi\hbar)^s} \int_{\mathbb{R}^{2s}} dr' F(r') \exp\left(-\|r' - r\|_{\alpha,\hbar}^2\right) \\ &= \frac{1}{\pi^s} \int_{\mathbb{R}^s \times \mathbb{R}^s} du dv F\left(q + u\sqrt{2\alpha\hbar}, p + v\sqrt{\frac{2\hbar}{\alpha}}\right) \exp\left(-(\|u\|^2 + \|v\|^2)\right), \end{aligned} \quad (13)$$

whence the classical limit

$$\lim_{\hbar \rightarrow 0} \gamma_{0,\hbar} \circ \gamma_{\hbar,0}[F](r) = F(r) \quad (14)$$

ensues. If the classical system evolves in time according to a Hamiltonian function  $H(q, p)$  then the anti-Wick quantization allows one to recover such an evolution from the quantized one when  $\hbar \rightarrow 0$ , the simplest situation occurs when  $H(q, p)$  corresponds to a quantized  $\hat{H} = \sum_j \omega_j \hat{a}_j^\dagger \hat{a}_j$ . In such a case, phase-space points  $r = (q, p)$  evolve into  $r_t = (q_t, p_t) = A_t r$  where  $A_t$  is an  $s \times s$  symplectic matrix; namely

$$\Omega A_t = A_{-t}^T \Omega, \quad (15)$$

where  $A_{-t}$ , respectively  $A_{-t}^T$  denote the inverse of the matrix  $A_t$ , respectively its transposed. Furthermore, exactly the same transformation affects the operators in  $\hat{r}$  when subjected to the quantized Hamiltonian  $\hat{H}$  while the state  $|0\rangle_\hbar$  does not change. Therefore, Weyl operators are sent into Weyl operators according to

$$\begin{aligned} \hat{W}(r) \mapsto \mathcal{U}_t[\hat{W}_\hbar(r)] &= \hat{U}_t \hat{W}_\hbar(r) \hat{U}_t^\dagger = e^{it\hat{H}/\hbar} \hat{W}_\hbar(r) e^{-it\hat{H}/\hbar} \\ &= \exp\left(\frac{i}{\hbar}(r, \Omega A_t \hat{r})\right) = \exp\left(\frac{i}{\hbar}(A_{-t} r, \Omega \hat{r})\right) = \hat{W}_\hbar(A_{-t} r). \end{aligned} \quad (16)$$

Then,

$$\begin{aligned} \hbar \langle z_r | \hat{U}_t | z_{r'} \rangle_\hbar &= \hbar \langle z_r | \hat{U}_t \hat{W}_\hbar(z_{r'}) \hat{U}_t^\dagger | 0 \rangle_\hbar \\ &= \hbar \langle z_r | z_{A_{-t} r'} \rangle_\hbar, \end{aligned} \quad (17)$$

so that

$$\gamma_{0,\hbar} \circ \mathcal{U}_t[\gamma_{\hbar,0}[F]](r) = \int_{\mathbb{R}^{2s}} dr' F_t(r') \exp\left(-\|r' - r\|_{\alpha,\hbar}^2\right) \quad \text{where } F_t(r') = F(A_t r'). \quad (18)$$

Then, in such a simple case, the classical limit of the quantum time-evolution amounts to the classical time-evolution:

$$\lim_{\hbar \rightarrow 0} \gamma_{0,\hbar} \circ \mathcal{U}_t[\gamma_{\hbar,0}[F]](r) = F_t(r) = F(A_t r). \quad (19)$$

### III. NONCOMMUTATIVE QUANTUM MECHANICS

We shortly review the formalism of noncommutative quantum mechanics, more details being available in [2]. We consider the two dimensional noncommutative configuration space, where the coordinates satisfy the commutation relation

$$[\hat{x}_i, \hat{x}_j] = i\theta\epsilon_{ij}, \quad (20)$$

with  $\theta$  a real positive parameter and  $\epsilon_{i,j}$  the completely antisymmetric tensor with  $\epsilon_{1,2} = 1$ . Since, the operators

$$b = \frac{1}{\sqrt{2\theta}}(\hat{x}_1 + i\hat{x}_2), \quad b^\dagger = \frac{1}{\sqrt{2\theta}}(\hat{x}_1 - i\hat{x}_2) \quad (21)$$

satisfy the commutation relations  $[b, b^\dagger] = 1$ , one can introduce a Fock-like vacuum vector  $|0\rangle$  such that  $b|0\rangle = 0$  and construct a non-commutative configuration space isomorphic to the boson Fock space

$$\mathcal{H}_c = \text{span}\{|n\rangle \equiv \frac{1}{\sqrt{n!}}(b^\dagger)^n|0\rangle\}_{n=0}^{n=\infty}, \quad (22)$$

where the span is taken over the field of complex numbers.

A proper Hilbert space over such non-commutative configuration space is the Hilbert-Schmidt Banach algebra  $\mathcal{H}_q$  of bounded operators  $\psi(\hat{x}_1, \hat{x}_2) \in \mathcal{B}(\mathcal{H}_c)$  on  $\mathcal{H}_c$  such that

$$\text{tr}_c(\psi(\hat{x}_1, \hat{x}_2)^\dagger \psi(\hat{x}_1, \hat{x}_2)) < \infty. \quad (23)$$

The  $\text{tr}_c$  denotes the trace over non-commutative configuration space and  $\mathcal{B}(\mathcal{H}_c)$  the set of bounded operators on  $\mathcal{H}_c$ . This space has a natural inner product and norm

$$(\phi(\hat{x}_1, \hat{x}_2), \psi(\hat{x}_1, \hat{x}_2)) = \text{tr}_c(\phi(\hat{x}_1, \hat{x}_2)^\dagger \psi(\hat{x}_1, \hat{x}_2)) \quad (24)$$

Next we introduce the non-commutative Heisenberg algebra

$$[\hat{X}_i, \hat{P}_j] = i\hbar\delta_{i,j}, \quad [\hat{X}_i, \hat{X}_j] = i\theta\epsilon_{i,j}, \quad [\hat{P}_i, \hat{P}_j] = 0, \quad (25)$$

where a unitary representation in terms of the operators  $\hat{X}_i$  and  $\hat{P}_i$  acting on the quantum Hilbert space (23) with the inner product (24) is

$$\hat{X}_i\psi(\hat{x}_1, \hat{x}_2) = \hat{x}_i\psi(\hat{x}_1, \hat{x}_2), \quad \hat{P}_i\psi(\hat{x}_1, \hat{x}_2) = \frac{\hbar}{\theta}\epsilon_{i,j}[\hat{x}_i, \psi(\hat{x}_1, \hat{x}_2)]. \quad (26)$$

In the above representation, the position acts by left multiplication and the momentum adjointly. We shall also consider the system to be equipped with a harmonic oscillator like Hamiltonian operator

$$\hat{H} = \sum_{i=1}^2 \left( \frac{1}{2m}\hat{P}_i^2 + \frac{1}{2}m\omega^2\hat{X}_i^2 \right), \quad (27)$$

and refer to the model as two non-interacting non-commutative quantum oscillators.

One can associate to position and momentum operators creation and annihilation-like operators  $\hat{A}_i, \hat{A}_i^\dagger$ ,  $i = 1, 2$  that satisfy the algebra

$$[\hat{A}_i, \hat{A}_j^\dagger] = \delta_{ij}; \quad [\hat{A}_i, \hat{A}_j] = 0. \quad (28)$$

The explicit expressions of the  $\hat{A}_i^\#$  are as follows [2]

$$\hat{A}_1 = \frac{1}{\sqrt{K_+}} \left( -\frac{\lambda_+}{\hbar}\hat{X}_1 - i\hat{P}_1 - i\frac{\lambda_+}{\hbar}\hat{X}_2 + \hat{P}_2 \right), \quad \hat{A}_1^\dagger = \frac{1}{\sqrt{K_+}} \left( -\frac{\lambda_+}{\hbar}\hat{X}_1 + i\hat{P}_1 + i\frac{\lambda_+}{\hbar}\hat{X}_2 + \hat{P}_2 \right) \quad (29)$$

$$\hat{A}_2 = \frac{1}{\sqrt{K_-}} \left( \frac{\lambda_-}{\hbar}\hat{X}_1 + i\hat{P}_1 - i\frac{\lambda_-}{\hbar}\hat{X}_2 + \hat{P}_2 \right), \quad \hat{A}_2^\dagger = \frac{1}{\sqrt{K_-}} \left( \frac{\lambda_-}{\hbar}\hat{X}_1 - i\hat{P}_1 + i\frac{\lambda_-}{\hbar}\hat{X}_2 + \hat{P}_2 \right), \quad (30)$$

where

$$\lambda_{\pm} = \frac{1}{2} \left( m\omega \sqrt{4\hbar^2 + m^2\omega^2\theta^2} \pm m^2\omega^2\theta \right), \quad K_{\pm} = \lambda_{\pm} \left( 4 \pm \frac{2\lambda_{\pm}\theta}{\hbar^2} \right). \quad (31)$$

Interestingly, the operators  $\hat{A}_j^{\#}$  can be interpreted as proper annihilation and creation operators as there is a vector in  $\mathcal{H}_q$ , that is a Hilbert-Schmidt operator  $\psi_0$  such that

$$\hat{A}_1|\psi_0\rangle = \hat{A}_2|\psi_0\rangle = 0, \quad (32)$$

given by [2]

$$\psi_0(\hat{x}_1, \hat{x}_2) = \exp\left(\frac{\beta}{2\theta}(\hat{x}_1^2 + \hat{x}_2^2)\right), \quad \beta = \ln(1 - \frac{\theta}{\hbar^2}\lambda_-) = -\ln(1 + \frac{\theta}{\hbar^2}\lambda_+). \quad (33)$$

After normalization, the ground state corresponding to  $|\psi_0\rangle$  is

$$|0, 0\rangle = \frac{|\psi_0\rangle}{\sqrt{\mathcal{N}}}, \quad \mathcal{N} = \frac{\hbar^4}{2\hbar^2\lambda_- - \theta\lambda_-^2}. \quad (34)$$

Furthermore, the Hamiltonian (27) becomes

$$\hat{H} = \frac{\lambda_+}{m} \hat{A}_1^\dagger \hat{A}_1 + \frac{\lambda_-}{m} \hat{A}_2^\dagger \hat{A}_2 + \frac{\lambda_+ + \lambda_-}{2m} \quad (35)$$

Clearly, there are two possible quantization and de-quantization schemes playing possibly together in this context: one is passing from a commutative to a non-commutative configuration space and back, another one is to pass from commuting position and momentum operators to non-commuting ones and back. In order to make the anti-Wick quantization works, we proceed by extending the coherent state construction of the previous section to this non-commutative quantum system with two degrees of freedom.

#### A. Gaussian-like states of the non-commutative quantum harmonic oscillators

In analogy with what we presented in Section (II), we introduce the coordinate vector  $r = (x_1, x_2, y_1, y_2)$  and the operator vector  $\hat{r} = (\hat{X}_1, \hat{X}_2, \hat{P}_1, \hat{P}_2)$ . Then, we construct the Weyl-like operators

$$\hat{W}_{\hbar,\theta}(r) = \exp\left(\frac{i}{\mu_{\hbar,\theta}}(r, \Omega \hat{r})\right), \quad (36)$$

where  $\mu_{\hbar,\theta}$  is a parameter with the dimension of an action. Using the commutation relations (25), the Weyl algebraic composition law (3) now read

$$\hat{W}_{\hbar,\theta}(r)\hat{W}_{\hbar,\theta}(r') = \exp\left(-\frac{i(\hbar + \theta)\mu_{\hbar,\theta}^2}{2}(r, \Omega r')\right) \hat{W}_{\hbar,\theta}(r + r'); \quad \Omega' = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix} \quad (37)$$

As to the parameter  $\mu_{\hbar,\theta}$ , it will eventually let vanish with  $\hbar \rightarrow 0$  and  $\theta \rightarrow 0$ . However, there are three possible ways we can reach the full commutative limit  $\hbar = 0 = \theta$ :

1. by linking  $\hbar$  and  $\theta$  so that one may consider the classical limit  $\mu_{\hbar,\theta} \rightarrow 0$ ;
2. by letting  $\theta \rightarrow 0$  first so to get to standard quantum mechanics and then let  $\hbar \rightarrow 0$ ;
3. by letting  $\hbar \rightarrow 0$  first so to get to a non-quantum non-commutative system and then let  $\theta \rightarrow 0$ .

In order to explore these three possibilities, we shall choose  $\mu_{\hbar,\theta}$  such that

$$\lim_{\theta \rightarrow 0} \mu_{\hbar,\theta} = \hbar; \quad \lim_{\hbar \rightarrow 0} \mu_{\hbar,\theta} = m\omega\theta. \quad (38)$$

Notice that the latter expression is the only natural constant with the dimensions of an action when  $\hbar = 0$  in the model. A most natural choice is provided by (31)

$$\mu_{\hbar,\theta} = \frac{\lambda_+}{m\omega} = \frac{\sqrt{4\hbar^2 + m^2\omega^2\theta^2} + m\omega\theta}{2}, \quad (39)$$

whereas  $\lambda_- \rightarrow 0$  when  $\hbar \rightarrow 0$ .

By inverting the relations (29) and (30),

$$\hat{X}_1 = \frac{\hbar}{2(\lambda_+ + \lambda_-)} \left( \sqrt{K_-}(A_2 + A_2^\dagger) - \sqrt{K_+}(A_1 + A_1^\dagger) \right) \quad (40)$$

$$\hat{X}_2 = -\frac{\hbar}{2i(\lambda_+ + \lambda_-)} \left( \sqrt{K_+}(A_1 - A_1^\dagger) + \sqrt{K_-}(A_2 - A_2^\dagger) \right) \quad (41)$$

$$\hat{P}_1 = \frac{1}{2i(\lambda_+ + \lambda_-)} \left( \lambda_+ \sqrt{K_-}(A_2 - A_2^\dagger) - \lambda_- \sqrt{K_+}(A_1 - A_1^\dagger) \right) \quad (42)$$

$$\hat{P}_2 = \frac{1}{2(\lambda_+ + \lambda_-)} \left( \lambda_+ \sqrt{K_-}(A_2 + A_2^\dagger) + \lambda_- \sqrt{K_+}(A_1 + A_1^\dagger) \right), \quad (43)$$

one can rewrite the Weyl operators (36) in the form

$$\hat{W}_{\hbar,\theta}(z_r) = \exp \left( z_{1,r} \hat{A}_1^\dagger + z_{2,r} \hat{A}_2^\dagger - z_{1,r}^* \hat{A}_1 - z_{2,r}^* \hat{A}_2 \right), \quad (44)$$

which is similar to (5), with  $z_r = (z_{1,r}, z_{2,r})$  a two dimensional complex vector whose real and imaginary parts are connected to the real four dimensional vector  $r$  by

$$\begin{pmatrix} \mathcal{R}e(z_{1,r}) \\ \mathcal{R}e(z_{2,r}) \\ \mathcal{I}m(z_{1,r}) \\ \mathcal{I}m(z_{2,r}) \end{pmatrix} = \hat{J}r; \quad \hat{J} = \frac{1}{2\mu_{\hbar,\theta}(\lambda_+ + \lambda_-)} \begin{pmatrix} \lambda_- \sqrt{K_+} & 0 & 0 & -\hbar \sqrt{K_+} \\ -\lambda_+ \sqrt{K_-} & 0 & 0 & -\hbar \sqrt{K_-} \\ 0 & \lambda_- \sqrt{K_+} & \hbar \sqrt{K_+} & 0 \\ 0 & \lambda_+ \sqrt{K_-} & -\hbar \sqrt{K_-} & 0 \end{pmatrix}. \quad (45)$$

By using the ground state (34) and the relations (28), we now introduce the non-commutative analogues of the coherent states (7),

$$|z_r\rangle_{\hbar,\theta} = \hat{W}_{\hbar,\theta}(z_r)|0,0\rangle = \exp \left( -\frac{\|z_r\|^2}{2} \right) \exp \left( z_{1,r} \hat{A}_1^\dagger + z_{2,r} \hat{A}_2^\dagger \right) |0,0\rangle, \quad (46)$$

where  $\|z_r\|^2 = |z_{1,r}|^2 + |z_{2,r}|^2$ . Exactly as in the case of (8), because of the algebraic relations (28), it follows that

$$\hat{A}_1 |z_r\rangle_{\hbar,\theta} = z_{1,r} |z_r\rangle_{\hbar,\theta}, \quad \hat{A}_2 |z_r\rangle_{\hbar,\theta} = z_{2,r} |z_r\rangle_{\hbar,\theta}. \quad (47)$$

These states are not exactly coherent states as they do not satisfy the non-commutative analog of minimal indeterminacy [4]; however, they have a Gaussian character and constitute an over-complete set.

**Lemma 1** *The states  $|z_r\rangle$  satisfy the resolution of identity*

$$\frac{1}{\pi^2} \int_{\mathbb{C}^2} dz_r |z_r\rangle_{\hbar,\theta} \langle z_r| = \frac{J}{\pi^2} \int_{\mathbb{R}^4} dr |z_r\rangle_{\hbar,\theta} \langle z_r|, \quad (48)$$

where  $J = \text{Det} \hat{J} = \frac{\hbar^2}{4\mu_{\hbar,\theta}^4} \hat{1}$  with  $\hat{J}$  the transformation matrix in (45).

**Proof:** Denote the integral by  $\hat{I}$ ; then, one checks whether  $\langle n_1, n_2 | \hat{I} | m_1, m_2 \rangle = \delta_{n_1, m_1} \delta_{n_2, m_2}$ , where the states

$$|n_1, n_2\rangle = \frac{(\hat{A}_1^\dagger)^{n_1} (\hat{A}_2^\dagger)^{n_2}}{\sqrt{n_1! n_2!}} |0, 0\rangle$$

constitute an orthonormal basis in the non-commutative Hilbert space  $\mathcal{H}_q$ . Then, (47) yields

$$\langle n_1, n_2 | \hat{I} | m_1, m_2 \rangle = (z_{1,r}^*)^{n_1} (z_{2,r}^*)^{n_2} (z_{1,r})^{m_1} (z_{2,r})^{m_2} e^{-\|z_r\|^2},$$

whence the result follows by Gaussian integration.

#### IV. THE CLASSICAL LIMITS OF THE NON-COMMUTATIVE HARMONIC OSCILLATORS

Following the prescriptions of the anti-Wick quantization in Section II, we start by choosing the classical algebra, that we choose as  $C_\infty(\mathbb{R}^4)$  made of continuous functions that vanish at infinity augmented with the identity function. Then, following (11) and (12), we define the quantization map de-quantization maps.

**Definition 1** Let  $\mathcal{W}_{\hbar, \theta}$  be the  $C^*$  algebra generated by the Weyl operators (36), the quantization of  $F \in C_\infty(\mathbb{R}^4)$  will be given by the positive unital map  $\gamma_{(\hbar, \theta), 0} : C_\infty(\mathbb{R}^4) \mapsto \mathcal{W}_{\hbar, \theta}$  defined by

$$C_\infty(\mathbb{R}^4) \ni F \mapsto \gamma_{(\hbar, \theta), 0}[F] =: \hat{F}_{\hbar, \theta} \in \mathcal{W}_{\hbar, \theta}, \quad \hat{F}_{\hbar, \theta} = \frac{J}{\pi^2} \int_{\mathbb{R}^4} dr F(r) |z_r\rangle_{\hbar, \theta} \langle z_r|, \quad (49)$$

while the de-quantization map by the following positive, unital map  $\gamma_{0, (\hbar, \theta)} : \mathcal{W}_{\hbar, \theta} \mapsto C_\infty(\mathbb{R}^4)$

$$\mathcal{W}_{\hbar, \theta} \ni \hat{X} \mapsto \gamma_{0, (\hbar, \theta)}[\hat{X}] =: X(r) \in C_\infty(\mathbb{R}^4), \quad X(r) = {}_{\hbar, \theta} \langle z_r | \hat{X} | z_r \rangle_{\hbar, \theta}. \quad (50)$$

In order to study the classical limit of the non-commutative quantum oscillators we shall focus upon the following functions

$$C_\infty(\mathbb{R}^4) \ni F \mapsto F_{\hbar, \theta} = \gamma_{0, (\hbar, \theta)} \circ \gamma_{(\hbar, \theta), 0}[F] \in C_\infty(\mathbb{R}^4) \quad (51)$$

that, after some manipulations reported in the Appendix, explicitly reads

$$\begin{aligned} F_{\hbar, \theta}(r) &= \frac{J}{\pi^2} \int_{\mathbb{R}^4} dr' F(r') |\langle z_r | z_{r'} \rangle|^2 \\ &= \frac{1}{\pi^2} \int_{\mathbb{R}^4} dw e^{-\|w\|^2} F\left(x_1 + f(w_1, w_2), x_2 + f(w_3, w_4), y_1 + g(w_3, w_4), y_2 - g(w_1, w_2)\right), \end{aligned} \quad (52)$$

where

$$f(x, y) = \frac{\mu_{\hbar, \theta} \sqrt{4\hbar^2 + m^2 \omega^2 \theta^2}}{2\sqrt{m\omega\hbar}} \left( \frac{x}{\sqrt{\gamma_+}} + \frac{y}{\sqrt{\gamma_-}} \right), \quad g(x, y) = \frac{\mu_{\hbar, \theta} \sqrt{m\omega} \sqrt{4\hbar^2 + m^2 \omega^2 \theta^2}}{\sqrt{4\hbar^2 + 2m^2 \omega^2 \theta^2}} \left( \frac{x}{\sqrt{\gamma_+}} - \frac{y}{\sqrt{\gamma_-}} \right), \quad (53)$$

$$\gamma_\pm = \frac{1}{2} \left( 1 \pm \frac{m\omega\theta}{\sqrt{4\hbar^2 + 2m^2 \omega^2 \theta^2}} \right). \quad (54)$$

In the following, we compute and discuss various possible limits in terms of  $\hbar$  and  $\theta$  or both going to zero.

##### A. Classical limit: $\mu_{\hbar, \theta} \rightarrow 0$

If  $\hbar$  and  $\theta$  vanish together with the same speed, that is if  $\hbar = \alpha \theta$ , with  $\alpha$  a suitable constant, then  $\mu_{\hbar, \theta} \simeq \hbar$ ,  $\gamma_\pm$  tend to constants and we get the classical limit

$$\lim_{\hbar \rightarrow 0} F_{\hbar, \theta}(r) = \frac{1}{\pi^2} \int_{\mathbb{R}^4} dw e^{-\|w\|^2} F(r) = F(r). \quad (55)$$

### B. Commutative configuration-space limit: $\theta \rightarrow 0$

In the limit  $\theta \rightarrow 0$ ,  $\mu_{\hbar,\theta} \rightarrow \hbar$ , from (53) and (54) we get the limit behaviours

$$\gamma_{\pm} = \frac{1}{2}, \quad f(x, y) = \sqrt{\frac{\hbar}{m\omega}}(x + y), \quad g(x, y) = \sqrt{\hbar m\omega}(x - y),$$

so that

$$\lim_{\theta \rightarrow 0} F_{\hbar,\theta}(r) = \frac{1}{\pi^2} \int_{\mathbb{R}^4} dw e^{-\|w\|^2} \times \\ \times F \left( x_1 + \sqrt{\frac{\hbar}{m\omega}}(w_1 + w_2), x_2 + \sqrt{\frac{\hbar}{m\omega}}(w_3 + w_4), y_1 + \sqrt{\hbar m\omega}(w_3 - w_4), y_2 + \sqrt{\hbar m\omega}(w_2 - w_1) \right). \quad (56)$$

This is nothing but the map (13) for two independent harmonic oscillators with  $\alpha = (m\omega)^{-1}$ , in fact by a change of variable that we include in the Appendix, we show that the equation (56) is equivalent to

$$F_{\hbar}(r) = \lim_{\theta \rightarrow 0} F_{\hbar,\theta}(r) = \frac{1}{\pi^2} \int_{\mathbb{R}^4} dudv e^{-u^2 - v^2} \times \\ \times F \left( x_1 + \sqrt{\frac{2\hbar}{m\omega}}u_1, x_2 + \sqrt{\frac{2\hbar}{m\omega}}u_2, y_1 + \sqrt{2\hbar m\omega}v_1, y_2 + \sqrt{2\hbar m\omega}v_2 \right). \quad (57)$$

The corresponding Weyl operators are

$$\hat{W}_{\hbar}(r) = \exp \left( \frac{i}{\hbar}(r, \Omega \hat{r}) \right),$$

and the Gaussian ground state

$$\psi_0(x_1, x_2) = \sqrt[4]{\frac{m\omega}{\pi\hbar}} \exp \left( -\frac{m\omega}{2\hbar}(x_1^2 + x_2^2) \right),$$

in the  $\hat{x}_{1,2}$  position representation. The classical limit  $\hbar \rightarrow 0$  then yields

$$\lim_{\hbar \rightarrow 0} F_{\hbar}(r) = F(r), \quad (58)$$

exactly as in the previous Section.

### C. Non-Commutative configuration-space limit: $\hbar \rightarrow 0$

In the limit  $\hbar \rightarrow 0$ ,  $\mu_{\hbar,\theta} \rightarrow m\omega\theta$ , from (53) and (53) we get the limit behaviours

$$\gamma_{\pm} = \frac{1}{2\sqrt{2}}(\sqrt{2} \pm 1), \quad f(x, y) \rightarrow +\infty \quad \text{almost everywhere on } \mathbb{R}^2 \\ g(x, y) = m\omega\sqrt{\theta} \left( \frac{x}{\sqrt{1 + \frac{1}{\sqrt{2}}}} - \frac{y}{\sqrt{1 - \frac{1}{\sqrt{2}}}} \right),$$

so that

$$\lim_{\hbar \rightarrow 0} F_{\hbar,\theta}(r) = \frac{1}{\pi^2} \int_{\mathbb{R}^4} dw e^{-\|w\|^2} \times \\ \times F_{\infty} \left( y_1 + m\omega\sqrt{\theta} \left( \frac{w_3}{\sqrt{1 + \frac{1}{\sqrt{2}}}} - \frac{w_4}{\sqrt{1 - \frac{1}{\sqrt{2}}}} \right), y_2 + m\omega\sqrt{\theta} \left( \frac{w_2}{\sqrt{1 - \frac{1}{\sqrt{2}}}} - \frac{w_1}{\sqrt{1 + \frac{1}{\sqrt{2}}}} \right) \right), \quad (59)$$



where the function  $F_\infty(y_1, y_2)$  denotes the limit  $\lim_{x_{1,2} \rightarrow +\infty} F(r)$ . Such limit exists and it is not trivial, in general, because the  $C^*$  algebra  $C_\infty(\mathbb{R}^4)$  contains also functions of the form

$$(f_1(x_1) + c_1)(f_2(x_2) + c_2)g_1(y_1)g_2(y_2) ,$$

where  $c_i$  are constants and  $f_i(x_i)$ ,  $i = 1, 2$ , vanish when their arguments go to  $\pm\infty$ .

The resulting expression coincides with the map (13) for the case of a commutative  $C^*$  algebra of functions  $C_\infty(\mathbb{R}^2)$ , Weyl operators of the form

$$\hat{W}_\theta(r) = \exp\left(\frac{i}{m\omega\theta}(y_1\hat{x}_2 - y_2\hat{x}_1)\right) , \quad [\hat{x}_1, \hat{x}_2] = i\theta , r = (y_1, y_2) \quad (60)$$

and Gaussian ground state

$$\psi_0(x_1) = \sqrt[4]{\frac{1}{\pi\theta}} \exp\left(-\frac{x_1^2}{2\theta}\right) ,$$

in the  $\hat{x}_1$ -representation where  $(\hat{x}_2\psi)(x_1) = -i\theta\psi'(x_1)$ . Under suitable change of variables, (59) is equivalent to

$$F_\theta(y_1, y_2) = \lim_{\hbar \rightarrow 0} F_{\hbar, \theta}(r) = \frac{1}{\pi} \int_{\mathbb{R} \times \mathbb{R}} dv e^{-\|v\|^2} F_\infty(y_1 + m\omega\sqrt{2\theta}v_1, y_2 + m\omega\sqrt{2\theta}v_2) . \quad (61)$$

This is the expression (13) with  $\hbar$  substituted by  $m\omega\theta$  and  $\alpha = (m\omega)^{-1}$ .

Then, in analogy with Section II, one defines two positive maps. The first map is a configuration space quantization map  $\gamma_{\theta,0} : C_\infty(\mathbb{R}^2) \mapsto \mathcal{W}_\theta$  from the  $C^*$  algebra of continuous functions over  $\mathbb{R}^2$  which vanish at infinity equipped with the identity function into the  $C^*$  algebra generated by the Weyl operators (60),

$$C_\infty(\mathbb{R}^2) \ni F \mapsto \gamma_{\theta,0}[F] =: \hat{F}_\theta \in \mathcal{W}_\theta , \quad \hat{F}_\theta = \frac{1}{2\pi m\omega\theta} \int_{\mathbb{R}^2} dr F(r) |z_r\rangle_{\theta\theta} \langle z_r| , \quad (62)$$

where

$$|z_r\rangle_\theta = \exp\left(\frac{i}{m\omega\theta}(y_1\hat{x}_2 - y_2\hat{x}_1)\right)|\psi_0\rangle_\theta , \quad z_r = -\frac{1}{m\omega\sqrt{2\theta}}(y_1 + iy_2) . \quad (63)$$

The second map is a de-quantizing configuration space map  $\gamma_{0,\theta} : \mathcal{W}_\theta \mapsto C_\infty(\mathbb{R}^2)$  given by

$$\mathcal{W}_\theta \ni \hat{X} \mapsto \gamma_{0,\theta}[\hat{X}] \in C_\infty(\mathbb{R}^2) , \quad X(r) = {}_\theta\langle z_r | \hat{X} | z_r \rangle_\theta , \quad (64)$$

which *de-quantizes* the operator  $\hat{X}$  mapping it back to a function in  $C_\infty(\mathbb{R}^2)$ . By combining the two maps, one finds that  $\gamma_{0,\theta} \circ \gamma_{\theta,0}[F](r)$  equals (61).

By letting  $\theta \rightarrow 0$ , one removes the non-commutativity of the configuration space and get back to a continuous function, on  $\mathbb{R}^2$  instead of  $\mathbb{R}^4$ :

$$\lim_{\theta \rightarrow 0} F_\theta(r) = F_\infty(y_1, y_2) . \quad (65)$$

We thus see that removal of quantum non-commutativity followed by removal of configuration space non-commutativity does not get back to the initial commutative algebra of continuous functions over  $\mathbb{R}^4$ , but on "half" space. Therefore, the two de-quantizing limits do not commute:

$$\lim_{\theta \rightarrow 0} \lim_{\hbar \rightarrow 0} \neq \lim_{\hbar \rightarrow 0} \lim_{\theta \rightarrow 0} . \quad (66)$$

In the next section we study how this non-exchangeability of limits affects as simple a time-evolution as the one generated by the Hamiltonian (27).

## V. CLASSICAL LIMIT OF THE NON-COMMUTATIVE TIME EVOLUTION

We now consider the time-evolution generated by the Hamiltonian (27), using as dimensional action, not  $\hbar$ , but the parameter  $\mu_{\hbar,\theta}$  in (39). The unitary time-evolutor on the non-commutative Hilbert space  $\mathcal{H}_q$  is thus given by

$$\hat{U}_t = \exp \left( - \frac{it}{\mu_{\hbar,\theta}} \hat{H} \right). \quad (67)$$

Its action on the Weyl operators in the forms (36) and (44) is easily computed to be

$$\begin{aligned} \hat{U}_t^\dagger \hat{W}_{\hbar,\theta}(z) \hat{U}_t &= \exp \left( e^{\frac{it}{m\mu_{\hbar,\theta}} \lambda_+} z_1 \hat{A}_1^\dagger + e^{\frac{it}{m\mu_{\hbar,\theta}} \lambda_-} z_2 \hat{A}_2^\dagger - e^{-\frac{it}{m\mu_{\hbar,\theta}} \lambda_+} \bar{z}_1 \hat{A}_1 - e^{-\frac{it}{m\mu_{\hbar,\theta}} \lambda_-} \bar{z}_2 \hat{A}_2 \right) \\ &= \exp \left( r, \Omega A_{t,\hbar,\theta} \hat{r} \right) = \exp \left( A_{-t,\hbar,\theta} r, \Omega \hat{r} \right) = \hat{W}_{\hbar,\theta}(r_{-t}), \end{aligned} \quad (68)$$

where, from symplecticity  $\Omega A_{t,\hbar,\theta} = A_{-t,\hbar,\theta}^T \Omega$ ,

$$r_{-t} = A_{-t,\hbar,\theta} r, \quad A_{t,\hbar,\theta} = \begin{pmatrix} \cos \omega_+ t & 0 & -\sin \omega_+ t & 0 \\ 0 & \cos \omega_- t & 0 & -\sin \omega_- t \\ \sin \omega_+ t & 0 & \cos \omega_+ t & 0 \\ 0 & \sin \omega_- t & 0 & \cos \omega_- t \end{pmatrix}, \quad (69)$$

where the oscillation frequencies are given by

$$\omega_\pm = \frac{\lambda_\pm}{m\mu_{\hbar,\theta}} = \frac{(m\omega\sqrt{4\hbar^2 + m^2\omega^2\theta^2} \pm m^2\omega^2\theta)}{2m\mu_{\hbar,\theta}}. \quad (70)$$

Since the ground state  $|0,0\rangle$  in (34) is left invariant by  $U_t$ , one finds that the time-evolution of the quantized function in (49) is given by

$$\hat{F}_{\hbar,\theta}(t) = \hat{U}_t^\dagger F_{\hbar,\theta} \hat{U}_t \in \mathcal{W}_{\hbar,\theta} = \frac{J}{\pi^2} \int_{\mathbb{R}^4} dr F(r) |z_r(-t)\rangle_{\hbar,\theta\hbar,\theta} \langle z_r(-t)| = \frac{J}{\pi^2} \int_{\mathbb{R}^4} dr F_t(r) |z_r\rangle_{\hbar,\theta\hbar,\theta} \langle z_r|, \quad (71)$$

where it has been used that  $\text{Det}(A_{t,\hbar,\theta}) = 1$  and has been set  $F_t(r) = F(A_{t,\hbar,\theta} r)$ . Then, (51) yields

$$\begin{aligned} F_{\hbar,\theta,t}(r) &= \gamma_{0,(\hbar,\theta)} \left[ \hat{U}_t^\dagger \gamma_{(\hbar,\theta),0} [F] \hat{U}_t \right](r) = \frac{J}{\pi^2} \int_{\mathbb{R}^4} dr' F_t(r') |\langle z_r | z_{r'} \rangle|^2 \\ &= \frac{1}{\pi^2} \int_{\mathbb{R}^4} dw e^{-\|w\|^2} F_t(r + h(w)), \quad h(w) = \left( f(w_1, w_2), f(w_3, w_4), g(w_3, w_4), -g(w_1, w_2) \right), \end{aligned} \quad (72)$$

with the functions  $f, g$  as in (53).

### A. Classical limit: $\mu_{\hbar,\theta} \rightarrow 0$

If  $\hbar$  and  $\theta$  vanish together with the same speed, that is if  $\hbar = \alpha \theta$ , with  $\alpha$  a suitable constant, then  $\mu_{\hbar,\theta} \simeq \hbar$ ,  $\gamma_\pm$  tend to constants and we get the classical limit as in the time independent case

$$\lim_{\hbar \rightarrow 0} F_{t,\hbar,\theta}(r) = \frac{1}{\pi^2} \int_{\mathbb{R}^4} dw e^{-\|w\|^2} F_t(r) = F_t(r). \quad (73)$$

### B. Commutative configuration-space limit: $\theta \rightarrow 0$

By letting  $\theta \rightarrow 0$  in (72) and thus recovering the commutative quantum mechanics context, from (70) one has  $\lim_{\theta \rightarrow 0} \omega_{\pm} = \omega$  while for the evolution matrix in (68)

$$A_t = \lim_{\theta \rightarrow 0} A_{t,\hbar,\theta} = \begin{pmatrix} \cos \omega t & 0 & -\sin \omega t & 0 \\ 0 & \cos \omega t & 0 & -\sin \omega t \\ \sin \omega t & 0 & \cos \omega t & 0 \\ 0 & \sin \omega t & 0 & \cos \omega t \end{pmatrix}. \quad (74)$$

Then, in this limit one gets

$$F_{t,\hbar}(r) = \lim_{\theta \rightarrow 0} F_{t,\hbar,\theta}(r) = \frac{1}{\pi^2} \int_{\mathbb{R}^4} dw e^{-\|w\|^2} F(A_{-t}(r + h(w))). \quad (75)$$

This corresponds to the commutative quantum mechanical time evolution of two identical identical independent harmonic oscillators.

In the classical limit  $\hbar \rightarrow 0$  one obviously recovers the time-evolution of two classical harmonic oscillators whose canonical coordinates evolve according to the symplectic matrix (74):

$$\lim_{\hbar \rightarrow 0} F_{t,\hbar}(r) = F(A_{-t} r). \quad (76)$$

### C. Non-commutative configuration limit: $\hbar \rightarrow 0$

By letting  $\hbar \rightarrow 0$  in (72) and thus going to the non-commutative configuration space context, from (70) one has  $\lim_{\theta \rightarrow 0} \omega_+ = \omega$ , while  $\lim_{\theta \rightarrow 0} \omega_- = 0$ ; thus, for the evolution matrix in (68)

$$B_t = \lim_{\hbar \rightarrow 0} A_{t,\hbar,\theta} = \begin{pmatrix} \cos \omega t & 0 & -\sin \omega t & 0 \\ 0 & 1 & 0 & 0 \\ \sin \omega t & 0 & \cos \omega t & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}. \quad (77)$$

The previous matrix cannot be used directly in performing the limit in in (72); indeed, we have to take into account that  $A_{t,\hbar,\theta}$  mixes the components of the vector  $r + h(w)$  and the function  $f(x, y)$  diverges as  $1/\hbar$ . However, when  $f(x, y)$  multiplies  $\sin \omega_- t$  the product vanishes since  $\omega_- \simeq \hbar^2$ . Therefore, when  $\hbar \rightarrow 0$ , from (59), one gets

$$\begin{aligned} F_{t,\theta}(y_2) &= \lim_{\hbar \rightarrow 0} F_{t,\hbar,\theta}(r) = \frac{1}{\pi^2} \int_{\mathbb{R}^4} dw e^{-\|w\|^2} F_{\infty} \left( y_2 + m\omega\sqrt{\theta} \left( \frac{w_2}{\sqrt{1 - \frac{1}{\sqrt{2}}}} - \frac{w_1}{\sqrt{1 + \frac{1}{\sqrt{2}}}} \right) \right) \\ &= \frac{1}{\sqrt{\pi}} \int_{\mathbb{R}} dv_2 e^{-v_2^2} F_{\infty}(y_2 + m\omega\sqrt{2\theta}v_2), \end{aligned} \quad (78)$$

where the function  $F_{\infty}(y_2)$  denotes the limit  $\lim_{x_1, x_2, y_1 \rightarrow +\infty} F(r)$  and is effectively a function of  $y_{1,2}$  only while  $r = (x_1, x_2, y_1, y_2)$ . The only footprint of the non-commutative quantum dynamics is the reduction of the dependence of the initial continuous functions from  $r \in \mathbb{R}^4$  to  $y_2 \in \mathbb{R}$ . Indeed, the full classical limit yields

$$\lim_{\theta \rightarrow 0} F_{t,\theta}(y_2) = F_{\infty}(y_2). \quad (79)$$

Therefore, starting with the continuous functions over  $\mathbb{R}^4$ , letting the dynamics act and then removing the standard non-commutativity before removing the configuration space non-commutativity one loses track of the time-evolution and even reduces, after the complete classical limit, the domain of definition of the continuous functions from  $\mathbb{R}^4$  to  $\mathbb{R}$ .

## VI. CONCLUSION

We have considered the classical limit of two independent quantum harmonic oscillators, with  $\hbar$  as quantization parameter, whose position coordinates are themselves non-commuting operators, with non-commutative deformation parameter  $\theta$ . This non-commutative quantum model allows for the construction of creation and annihilation operators with a corresponding Weyl algebra; we have thus studied the classical limit by means of the so-called anti-Wick quantization scheme that uses coherent states to map a commutative  $C^*$  algebra of continuous functions into the non-commutative  $C^*$  algebra generated by the Weyl operators and to map these operators back to continuous functions.

Three possibilities appear to implement the scheme:

1. to link  $\hbar$  and  $\theta$  so that one may consider the classical limit  $\mu_{\hbar,\theta} \rightarrow 0$ ;
2. let  $\hbar \rightarrow 0$  first so to get to a non-quantum non-commutative system and then let  $\theta \rightarrow 0$ ;
3. to let  $\theta \rightarrow 0$  first so to get to standard quantum mechanics and then let  $\hbar \rightarrow 0$ .

In the given model, the first possibility corresponds to an anti-Wick quantization procedure which quantizes a  $C^*$  algebra of continuous functions over  $\mathbb{R}^4$  and de-quantizes it back to the same algebra. In the second case, when  $\theta \rightarrow 0$ , one gets the Weyl algebra of two standard quantum oscillators and then the continuous functions over  $\mathbb{R}^4$  when  $\hbar \rightarrow 0$ . Instead, the third possibility is such that  $\hbar \rightarrow 0$  first yields a quantization scheme of a  $C^*$  algebra of continuous functions over  $\mathbb{R}^2$  (not  $\mathbb{R}^4$ ) and then  $\theta \rightarrow 0$  maps the Weyl algebra generated by the non-commuting position coordinates of the two oscillators back to the continuous functions over  $\mathbb{R}^2$ . The non-exchangeability of the two limits

$$\lim_{\theta \rightarrow 0} \lim_{\hbar \rightarrow 0} \neq \lim_{\hbar \rightarrow 0} \lim_{\theta \rightarrow 0} ,$$

becomes even more evident when one considers the dynamics of the non-commutative quantum oscillators generated by a quadratic Hamiltonian in the non-commutative quantum creations and annihilation operators. In this case, while the classical limit performed according to the first and second possibilities yields the classical Hamiltonian dynamics of two identical, independent harmonic oscillators, in the third case the non-commutative non-quantum dynamics does not survive the classical limit, but for the fact that it further reduces to  $\mathbb{R}$  the space of definition of continuous functions initially defined on  $\mathbb{R}^4$ .

## VII. APPENDIX

Let us consider

$$F_{\hbar,\theta}(r) = \frac{J}{\pi^2} \int_{\mathbb{R}^4} dr' F(r') |\langle z(r) | z(r') \rangle|^2 = \frac{\hbar^2}{4\mu_{\hbar,\theta}^4 \pi^2} \int_{\mathbb{R}^4} dr' F(r') e^{-E(r,r')} , \quad (80)$$

where

$$\begin{aligned} E(r, r') &= |z_1(r) - z_1(r')|^2 + |z_2(r) - z_2(r')|^2 \\ &= \frac{1}{4\mu_{\hbar,\theta}^2 (\lambda_+ + \lambda_-)^2} \times \left\{ (\lambda_-^2 K_+ + \lambda_+^2 K_-) (x_1 - x'_1)^2 + \hbar^2 (K_+ + K_-) (y_2 - y'_2)^2 \right. \\ &\quad + (\lambda_-^2 K_+ + \lambda_+^2 K_-) (x_2 - x'_2)^2 + \hbar^2 (K_+ + K_-) (y_1 - y'_1)^2 \\ &\quad \left. - 2\hbar (\lambda_- K_+ - \lambda_+ K_-) (x_1 - x'_1) (y_2 - y'_2) + 2\hbar (\lambda_- K_+ - \lambda_+ K_-) (x_2 - x'_2) (y_1 - y'_1) \right\} . \quad (81) \end{aligned}$$

First, by setting  $u_i = (x_i - x'_i)$  and  $v_i = (y_i - y'_i)$ ,  $i = 1, 2$ , one gets

$$F_{\hbar,\theta}(r) = \frac{\hbar^2}{4\mu_{\hbar,\theta}^4\pi^2} \int_{\mathbb{R}^4} du_1 du_2 dv_1 dv_2 F(x_1 + u_1, x_2 + u_2, y_1 + v_1, y_2 + v_2) e^{-D(u_1, u_2, v_1, v_2)} \quad (82)$$

$$D(u_1, u_2, v_1, v_2) = \frac{1}{4m\omega\mu_{\hbar,\theta}^2\sqrt{4\hbar^2 + m^2\omega^2\theta^2}} \left\{ 4m^2\omega^2\hbar^2 u_1^2 + (4\hbar^2 + 2m^2\omega^2\theta^2)v_2^2 - 4m^2\omega^2\hbar\theta u_1 v_2 \right. \quad (83)$$

$$\left. + 4m^2\omega^2\hbar^2 u_2^2 + (4\hbar^2 + 2m^2\omega^2\theta^2)v_1^2 + 4m^2\omega^2\hbar\theta u_2 v_1 \right\}. \quad (84)$$

Next, the change of variables  $\bar{u}_i = \frac{4m\omega\hbar}{\mu_{\hbar,\theta}} u_i$ ,  $\bar{v}_i = \frac{2\sqrt{4\hbar^2 + 2m^2\omega^2\theta^2}}{\mu_{\hbar,\theta}} v_i$ ,  $i = 1, 2$  yields

$$F_{\hbar,\theta}(r) = \frac{1}{16m^2\omega^2(4\hbar^2 + 2m^2\omega^2\theta^2)} \int_{\mathbb{R}^4} d\bar{u}_1 d\bar{u}_2 d\bar{v}_1 d\bar{v}_2 e^{-G(\bar{u}_1, \bar{u}_2, \bar{v}_1, \bar{v}_2)} \times \quad (85)$$

$$\times F\left(x_1 + \frac{\mu_{\hbar,\theta}}{2m\omega\hbar} \bar{u}_1, x_2 + \frac{\mu_{\hbar,\theta}}{2m\omega\hbar} \bar{u}_2, y_1 + \frac{\mu_{\hbar,\theta}}{\sqrt{4\hbar^2 + 2m^2\omega^2\theta^2}} \bar{v}_1, y_2 + \frac{\mu_{\hbar,\theta}}{\sqrt{4\hbar^2 + 2m^2\omega^2\theta^2}} \bar{v}_2\right) \quad (86)$$

$$G(\bar{u}_1, \bar{u}_2, \bar{v}_1, \bar{v}_2) = \frac{\gamma_+(\bar{u}_1 - \bar{v}_2)^2 + \gamma_-(\bar{u}_1 + \bar{v}_2)^2 + \gamma_+(\bar{u}_2 + \bar{v}_1)^2 + \gamma_-(\bar{u}_2 - \bar{v}_1)^2}{4m\omega\sqrt{4\hbar^2 + 2m^2\omega^2\theta^2}}, \quad (87)$$

with  $\gamma_{\pm} = \frac{1}{2} \left( 1 \pm \frac{m\omega\theta}{\sqrt{4\hbar^2 + 2m^2\omega^2\theta^2}} \right)$ .

The expression in equation (86) can be diagonalized by setting

$$\begin{aligned} w_1 &= \sqrt{\frac{4m\omega}{\gamma_+}} \sqrt[4]{4\hbar^2 + 2m^2\omega^2\theta^2} (\bar{u}_1 - \bar{v}_2), & w_2 &= \sqrt{\frac{4m\omega}{\gamma_-}} \sqrt[4]{4\hbar^2 + 2m^2\omega^2\theta^2} (\bar{u}_1 + \bar{v}_2) \\ w_3 &= \sqrt{\frac{4m\omega}{\gamma_+}} \sqrt[4]{4\hbar^2 + 2m^2\omega^2\theta^2} (\bar{u}_2 + \bar{v}_1), & w_4 &= \sqrt{\frac{4m\omega}{\gamma_-}} \sqrt[4]{4\hbar^2 + 2m^2\omega^2\theta^2} (\bar{u}_2 - \bar{v}_1) \end{aligned}$$

This finally yields

$$\begin{aligned} F_{\hbar,\theta}(r) &= \frac{1}{\pi^2} \int_{\mathbb{R}^4} dw_1 dw_2 dw_3 dw_4 e^{-(w_1^2 + w_2^2 + w_3^2 + w_4^2)} \times \quad (88) \\ &\times F\left(x_1 + \frac{\mu_{\hbar,\theta} \sqrt[4]{4\hbar^2 + m^2\omega^2\theta^2}}{2\sqrt{m\omega\hbar}} \left( \frac{w_1}{\sqrt{\gamma_+}} + \frac{w_2}{\sqrt{\gamma_-}} \right), x_2 + \frac{\mu_{\hbar,\theta} \sqrt[4]{4\hbar^2 + m^2\omega^2\theta^2}}{2\sqrt{m\omega\hbar}} \left( \frac{w_3}{\sqrt{\gamma_+}} + \frac{w_4}{\sqrt{\gamma_-}} \right), \right. \\ &\left. y_1 + \frac{\mu_{\hbar,\theta} \sqrt{m\omega} \sqrt[4]{4\hbar^2 + m^2\omega^2\theta^2}}{\sqrt{4\hbar^2 + 2m^2\omega^2\theta^2}} \left( \frac{w_3}{\sqrt{\gamma_+}} - \frac{w_4}{\sqrt{\gamma_-}} \right), y_2 + \frac{\mu_{\hbar,\theta} \sqrt{m\omega} \sqrt[4]{4\hbar^2 + m^2\omega^2\theta^2}}{\sqrt{4\hbar^2 + 2m^2\omega^2\theta^2}} \left( \frac{w_2}{\sqrt{\gamma_-}} - \frac{w_1}{\sqrt{\gamma_+}} \right) \right). \end{aligned}$$

Then, one obtains equation (57) by means of the following change of variables in equation (56):

$$u_1 = \frac{w_1 + w_2}{\sqrt{2}}, \quad u_2 = \frac{w_3 + w_4}{\sqrt{2}}, \quad v_1 = \frac{w_3 - w_4}{\sqrt{2}}, \quad v_2 = \frac{w_2 - w_1}{\sqrt{2}}. \quad (89)$$

The Jacobian for this change of variable is  $J = 1$  and  $w_1^2 + w_2^2 + w_3^2 + w_4^2 = u_1^2 + u_2^2 + v_1^2 + v_2^2$ .

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